مجلة الجامعة

The vital role of the theory of angular momentum in mathematics and physics

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Abstract

The theory of angular momentum (AM) plays a very important and vital role in applied mathematics and physics. In this paper, we revise this topic emphasizing the various commutation relations obeyed by its components, and its relation with Casimir operators. Hence, A good account on Lie algebra , which is supposed to be the algebra of commutators, is given stressing its relation with Casimir operators. As an obvious application in this concern , we turn our attention to angular momentum operators (AMO) $\boldsymbol{L}$, $\boldsymbol{L}^2$, supported with sample examples on the application of the Casimir operator $\boldsymbol{L}^2$ .Finally ,we give a short review of recent works on the subject of Casimir operators.

Keywords: Casimir Operators, Lie algebra, angular momentum .

1.Introduction

Lie algebra ,which deals with commutator algebra is one of the important subjects in applied mathematics and theoretical physics, and hence Casimir operators which rely on the theory of angular momentum. Accordingly this article came into light based on a previous work [1].

In section 2, we introduce Lie algebra and some applications ,where Schrodinger equation is revised. Angular momentum operators are discussed in the next section with their relation with solving Schrodinger equation in spherical coordinates [2].The commutation relations of the angular momentum operators are then given ,concentrating on Casimir operator $\boldsymbol{L}^2$ along with their relations with Lie algebra[3].

In section 4 , few applications are given supported with some examples. The spin angular momentum(SAM) is exposed to in section 5 , followed by the total angular momentum(TAM) and its importance in quantum mechanics [4].

Finally, we conclude, in the last section , with a short discussion on recent works on the subject [7][8][9].

2.Lie algebra

Lie algebra g is a vector space on the real field R with a binary operation as follows

$$[..,..]: g \times g \rightarrow g \tag{1}$$

$[..,..]$ Is called Lie bracket which satisfies the following properties

i-$[x, x] = 0$

ii-$[x + y, z] = [x, z] + [y, z]$

iii-$[x, y + z] = [x, y] + [x, z]$

iv-$[\alpha x, y] = \alpha[x, y]$,$\forall\ \alpha \epsilon R$

v-$\big[x, [y, z]\big] + \big[y, [z, x]\big] + \big[z, [x, y]\big] = 0$,$\forall\ x, y, z \,\epsilon\, g$

Property v is called Jacobi identity [2].

Note that Lie algebra is commutative if $[g, g] = 0$, i.e. $[x, y] = 0\ \forall\ x, y\ \epsilon\ g$.

2.1 Few examples

Example 1

The set of square matrices of dimension $(n \times n)$, $M_n(C)$, where C is set of complex numbers ,satisfies all properties cited above.

Example 2

If g on R ,and if D is the differential operator on g ,which satisfies

$$D\{x,y\} = [Dx,y] + [x,Dy] \qquad \forall\ x,y \,\epsilon\, g \qquad (2)$$

Then we get a new set which we represent as Der(g) ,with the two operations given below

$$(D + D_1)(x) = D(x) + D_1(x) \qquad (3)$$

And

$$(\alpha D)(x) = \alpha D(x)\ , \forall\ x \epsilon\ g\ and\ D, D_1\ \epsilon\ Der(g) \qquad (3)$$

We note that $Der(g)$ is also a Lie Algebra [1].

Example 3

g has the property $[x,y] = -[y,x]$,this can be seen from the fact that $[x+y,x+y] = 0$ [from (i) of the definition of Lie algebra]; hence one gets

$[x+y,x+y\} = [x,x] + [x,y] + [y,x] + [y,y] = 0 + [x,y] + [y,x] + 0 = [x,y] + [y,x] = 0 \rightarrow [x,y] = -[y,x]$,

Remarks

i-If g is a Lie Algebra, then $Der(g)$ is also a Lie Algebra [1].

ii-If for $D\ and\ D_1$,we define the binary operation o on $Der(g)$ as $[D,D_1] = DoD_1 - D_oD$,then $[D,D_1]g$ is defined; and $[D,D_1]\ \epsilon\ Der(g)$. [1]

2.2 Schrodinger equation

The time-dependent Schrodinger equation(TDSE) is given

$$i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r},t) + V(\vec{r},t)\Psi(\vec{r},t) \quad (4)$$

Where $\Psi(\vec{r},t)$is the wavefunction, $V(\vec{r},t)$ is the potential, m is the particle mass , $\vec{r}$ is the position vector, and $\hbar$ is Plank constant.

Separating the above equation, we get the time-independent Schrodinger equation(TISE) as

$$\mathbb{H}\emptyset = \boldsymbol{E}\emptyset \quad (5)$$

Where $\mathbb{H}$ is the Hamiltonian given by $\mathbb{H} = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(x,y,z)$; E the energy eigenvalue and $V(x,y,z)$ is the potential energy [5].

3.Angular momentum operators (AMO)

The TISE in spherical coordinates is written as

$$\nabla^2\emptyset(r,\theta,\varphi) + \frac{2m}{\hbar^2}(E - V)\emptyset(r,\theta,\varphi) = 0 \quad (6)$$

Where $\boldsymbol{\nabla}^2$ is the Laplacian operator given ,in spherical coordinates, by

$$\boldsymbol{\nabla}^2 \equiv \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \quad (7)$$

The angular part in Equation(7) is related to the total angular momentum as we will show in the paragraph to follow.

The classical angular momentum is defined as $\vec{L} = \vec{r} \times \vec{p}$, where $\vec{p}$ is the linear momentum; while in quantum mechanics $\vec{p} = -i\hbar\nabla$,and $\vec{L}$ in this case has the following components

$$\boldsymbol{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \boldsymbol{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \quad \boldsymbol{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \quad (8)$$

And the total angular momentum is

$$\boldsymbol{L}^2 = \boldsymbol{L}_x^2 + \boldsymbol{L}_y^2 + \boldsymbol{L}_z^2 = -\hbar^2\left(\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) \quad (9)$$

$\boldsymbol{L}^2$ is the so-called Casimir Operator.

Before we proceed, we ascertain that the AMO 's satisfy the following commutation relations

$$[\boldsymbol{L}_x, \boldsymbol{L}_y] = i\hbar\boldsymbol{L}_z \,, [\boldsymbol{L}_y, \boldsymbol{L}_z] = i\hbar\boldsymbol{L}_x \,, [\boldsymbol{L}_z, \boldsymbol{L}_x] = i\hbar\boldsymbol{L}_y \quad (10)$$

These commutation relations can be written in one formula as

$$[\boldsymbol{L}_l, \boldsymbol{L}_m] = i\hbar \in_{lmn} \boldsymbol{L}_n \,, l, m, n = 1,2,3. \quad (11)$$

$\{n, m, l\}$ *can be put into correspondence with* $\{x, y, z\}$,and $\in_{lmn}$ is the antisymmetric tensor [6].

3.1 A short account on commutators

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are two operators ,then the commutator of the two operators is defined as

$$[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{AB} - \boldsymbol{BA} \quad (12)$$

Commutators satisfy the followings

i-$[\boldsymbol{A}, \boldsymbol{B}] = -[\boldsymbol{B}, \boldsymbol{A}]$

ii-$[\boldsymbol{A}, \boldsymbol{A}^n] = 0 \;, n = 1,2,3, \ldots$

iii-$[k\boldsymbol{A}, \boldsymbol{B}] = [\boldsymbol{A}, k\boldsymbol{B}] = k[\boldsymbol{A}, \boldsymbol{B}]$

iv-$[\boldsymbol{A} + \boldsymbol{B}, \boldsymbol{C}] = [\boldsymbol{A}, \boldsymbol{C}] + [\boldsymbol{B}, \boldsymbol{C}]$; and $[\boldsymbol{A}, \boldsymbol{B} + \boldsymbol{C}] = [\boldsymbol{A}. \boldsymbol{B}] + [\boldsymbol{A}, \boldsymbol{C}]$

v-$[\boldsymbol{A}, \boldsymbol{BC}] = [\boldsymbol{A}, \boldsymbol{B}]\boldsymbol{C} + \boldsymbol{A}[\boldsymbol{B}, \boldsymbol{C}]$; and $[\boldsymbol{AB}, \boldsymbol{C}] = [\boldsymbol{A}, \boldsymbol{C}]\boldsymbol{B} + \boldsymbol{A}[\boldsymbol{B}, \boldsymbol{C}]$

Example 4

To compute $[\boldsymbol{x}, \boldsymbol{p}_x]$, we see that $[\boldsymbol{x}, \boldsymbol{p}_x] = \left[\boldsymbol{x}, -i\hbar\frac{\partial}{\partial x}\right] = i\hbar$ .In the same manner we can get $[\boldsymbol{x}, \boldsymbol{p}_x^2] = 2\hbar^2\frac{\partial}{\partial x}$.[1]

Example 5

The commutator built from the x-component of the linear momentum and the Hamiltonian is given by

$[\boldsymbol{p}_x, \mathbb{H}] = [\boldsymbol{p}_x, \boldsymbol{T} + \boldsymbol{V}] = \frac{1}{2m}([\boldsymbol{p}_x, \boldsymbol{p}_x^2] + [\boldsymbol{p}_x, \boldsymbol{p}_y^2] + [\boldsymbol{p}_x, \boldsymbol{p}_z^2]) + [\boldsymbol{p}_x, \boldsymbol{V}] =$
$[\boldsymbol{p}_x, \boldsymbol{V}] = [-i\hbar\frac{\partial}{\partial x}, \boldsymbol{V}] = -i\hbar\frac{\partial V}{\partial x}$ {1}.

Note that referring to property ii, we find that the AMO 's satisfy

$$\boldsymbol{L}_i, \boldsymbol{L}^2] = 0 \ , i = 1,2,3. \quad (13)$$

This will lead to the important result in quantum mechanics, which implies that the AMO's in Equation(13), have the same simultaneous eigenfunctions .Moreover, in solving quantum mechanical problems using spherical coordinates, the three commuting operators $\{\boldsymbol{L}_z, \boldsymbol{L}^2, \mathbb{H}\}$ are the operators to be dealt with.

3.2 Casimir operators and their relations with Lie algebra

It is clear that the AMO 's $\{\boldsymbol{L}_x, \boldsymbol{L}_y, \boldsymbol{L}_z\}$ are Lie operators from the first rank; if A is a Casimir operator, then it is assumed that

$$[\boldsymbol{A}, \boldsymbol{L}_i] = 0 \ , i = 1,2,3 \ ; [\boldsymbol{AB}, \boldsymbol{C}] = \boldsymbol{A}[\boldsymbol{B}, \boldsymbol{C}] + [\boldsymbol{A}, \boldsymbol{C}]\boldsymbol{B} \quad (14)$$

Now ,referring to Equation(11) and multiplying it with $\boldsymbol{L}$ once from right and once from left; moreover, with few mathematical manipulations we reach the result [1]

$$[\boldsymbol{L}, \boldsymbol{L}^2] = 0 \quad (15)$$

The last equation reflects the relationship of Casimir operator with the Lie operators(which are the AMO's).

4-Few applications

4.1 The magnetic and orbital quantum numbers

From Equation(15) , we can define a common eigenfunction $\mid \theta, \varphi >$ for the two operators $\boldsymbol{L}_z, \boldsymbol{L}^2$, and since $\boldsymbol{L}_z = -i\hbar\frac{\partial}{\partial\varphi}$, we get

$$\boldsymbol{L}_z \mid \theta, \varphi >= -i\hbar\frac{\partial|\theta,\varphi>}{\partial\varphi} = b \mid \theta, \varphi > \rightarrow b = m\hbar \ , m = 0,1,2,.. \quad (16)$$

And $\mid \theta,\varphi >= \frac{1}{\sqrt{2\pi}} e^{im\varphi} . m$ is called the magnetic quantum number (MQN) ; and we note that the bra, ket notations were used here to represent the wavefunctions.

On the same line it can be shown that [1][4]

$$\boldsymbol{L}^2 \mid \theta,\varphi > = \hbar^2 l(l+1) \mid \theta,\varphi >; l = 0,1,2,\dots,n-1 \tag{17}$$

Where l is the orbital angular quantum number (OAQN) and

$\mid \theta,\varphi >= Y_{l,m}(\theta,\varphi)$ is the spherical harmonic, given by

$$Y_{l,m}(\theta,\varphi) = (-1)^m \sqrt{\frac{(2l+1)(m-1)!}{4\pi(m+1)!}}\, p_l^m(\cos\theta) e^{im\varphi} \tag{18}$$

n is a quantum number related to the solution of the radial part of Schrodinger equation [4].

Example 6

If $\mid \theta,\varphi >=\mid m,l > = (6,-r) r e^{-r/3} \cos\theta$,then $\mid m,l > \equiv f(r) g(\theta)$; and since there is φ dependence $m = 0$ (this can be seen from the fact that $\boldsymbol{L}_z \mid \theta,\varphi >= 0$) ; $\boldsymbol{L}^2 \mid \theta,\varphi >= \hbar^2[2f(r)cos\theta]$ .Comparing this result with the general result $\boldsymbol{L}^2 \mid \theta,\varphi >= \hbar^2 l(l+1) \mid \theta,\varphi >$, we get $l = 1$ [1].

Example 7

If $\Psi_{lm} = \frac{1}{2\sqrt{6}}[3 \mid 0,0 > +2 \mid 1,1 > -\mid 1,0 > +\sqrt{10} \mid 1,-1 >]$, then to compute the eigenvalues of the two operators $\boldsymbol{L}_z$, $\boldsymbol{L}^2$, we see that [1]

$<\boldsymbol{L}_z >= -\frac{1}{4}\hbar$,$< \boldsymbol{L}^2 >= \frac{7}{12}\hbar^2$.

4.2 Raising and lowering operators

Two important operators are very much related to the forehead operators, namely the raising operator defined as

$$\boldsymbol{L}_+ = \boldsymbol{L}_x + i\boldsymbol{L}_y \tag{19}$$

And the lowering operator given by

$$L_- = L_x - iL_y \qquad (20)$$

Note that we can prove the following formulae [4][5]:-

i- $L_+L_- = L^2 - L_z^2 + \hbar L_z$.
ii- $[L_z, L_+] = \hbar L_+$
iii- Operating with L_z on $L_\pm \mid l, m >$, one gets

$(m \pm 1)\hbar L_\pm \mid l, m >$ respectively ; this shows the reason behind naming them as raising and lowering operators .Moreover,

$$L_\pm \mid l, m > = \hbar\sqrt{l(l+1) - m(m \pm 1)} \mid l, m+1 > .$$

Example 8

If $SO(3)$ is a Lie algebra composed of the three-dimensional matrices, then the three generators of this algebra are

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad , L_y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

And Casimir operator, in this case, is given as

$$L^2 = L_x^2 + L_y^2 + L_z^2 = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1(1+1)\mathbf{I}$$

$\mathbf{I}$ is the unit matrix ($l = 1$)[1].

5-Spin angular momentum

The spin angular momentum (SAM)($\hat{S}$) has to do with the spinning motion of the electrons in the atom; the electron has either sin up ,which is a rotational motion counterclockwise ,or spin down which is clockwise. The same rules ,obeyed by the orbital angular momentum (OAM), are also obeyed by SAM. e.g. we get $[S_x, S_y] = i\hbar\, S_z$; with similar forms for the other commutators.[1],[4].

For a particle ,having both OAM and SAM, it is said to have total angular momentum (TAM) ,and the TAM $\mathbf{J}$ is given by

$$\mathbf{J} = \boldsymbol{L} + \boldsymbol{S} \qquad (21)$$

Similar commutation relations are fulfilled by TAM operators [4].

Example 9

Since $\boldsymbol{S}_{\pm} \mid s, m_s >= \hbar\sqrt{s(s+1) - m_s(m_s \pm 1)} \mid s, m_s >$,as quoted before, and the state is in spin up mode($\alpha =\mid \frac{1}{2}, \frac{1}{2} >$,$s = \frac{1}{2}, m_s = 1/2$), then we have $\boldsymbol{S}_{\pm}\alpha = \hbar\alpha$ [1].

Example 10

We can also write the various AMO in a matrix form . To show that we prove that for the z-component of the SAM as follows[1]

Since the spin states are $\alpha =\mid \frac{1}{2}, \frac{1}{2} >$ and $\beta =\mid \frac{1}{2}, -\frac{1}{2} >$,and that the components of $\boldsymbol{S}_z$ are four, we get $(\boldsymbol{S}_z) = \begin{pmatrix} < \alpha \mid \hat{S}_z \mid \alpha > & < \alpha \mid \hat{S}_z \mid \beta > \\ < \beta \mid \hat{S}_z \mid \alpha > & < \beta \mid \hat{S}_z \mid \beta > \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{2} \times 1 & \frac{\hbar}{2} \times 0 \\ \frac{\hbar}{2} \times 0 & -\frac{\hbar}{2} \times 1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

6-Concluding discussion

The AMO ' s play a very important and vital role in quantum mechanics and hence in physics. Their relations with Lie algebra and Casimir operator made the study more attractive and fascinating ; computing the eingenfunctions and eigenvalues are few applications to mention.

Casimir operators are still a subject of research; for instance one of the somewhat recent topics in this concern is the computation of Casimir values for universal Lie Algebra, where it was shown that these values in the adjoint representation of simple Lie algebras can be expressed rationally in the universal Vogel's parameters α, β, γ [7]; moreover, the split Casimir operator for simple Lie algebras was given in conjunction with the cube of ad-representation and Vogel parameters [8].

A good and beautiful lecture was recently written by Daniel Bump, from Stanford University, on "The Casimir Operator " which covers most of the subjects that are related to it [9].

References

[1] Mohammed N.M. (2016) Casimir Operators and their applications, MSc. Thesis, University of Tripoli, Tripoli ,Libya.

[2] Lachello F.(2015) Lie Algebras and applications, 2nd Edition, Springer.

(pdf) ,National Academic digital Library.

http://ndl,ethernet.edu.et>8..PDF

[3] Snobl L.(2011) Representations of Lien algebras, Casimir operators and their applications, Lectures at the 5th Student Colloquium and School of Mathematical Physics, Stara, Lesna (29/8-4/9).

[4] Merzbacher E.(1998) Quantum Mechanics, 3rd Edition, john Wiley and Sons, New York.

[5] Ziock,K.(1975) Principles of quantum mechanics, John Wiley and Sons Incorporation ,New York.

[6] Awin,A.M.,Asharif, T.S.(2001) Tensors and their applications, 1st Edition ,Elga , Malta.

[7] Mkrtchyan R.L., Sergeev A.N.,Verselov A.P.(2011) Casimir values for universal Lie algebra , PDF

arKiv:1105.0115v2 [math-rt] 21/11/2011

https:// www.researchgate.net/publication/228912052

[8] Isaev A.P.,Krivonos S.O.,Provorov A.A.(2022)Split Casimir operator for simple Lie Algebras in the cube of ad-representation and Vogel parameters, PDF

arXiv:2212.14761v1 [math-ph]30/12/2022

[9] Daniel D.(2020) Lecture 6: The Casimir Operator,

Stanford University,

http:sporadic.stanford.edu.>…PDF